

DECOMPOSING ALMOST COMPLETE GRAPHS BY RANDOM TREES

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ABSTRACT. An old conjecture of Ringel states that every tree with m edges decomposes the complete graph K_{2m+1} . The best known lower bound for the order of a complete graph which admits a decomposition by every given tree with m edges is $O(m^3)$. We show that asymptotically almost surely a random tree with m edges and $p = 2m + 1$ a prime decomposes $K_{2m+1}(r)$ for every $r \geq 2$, the graph obtained from the complete graph K_{2m+1} by replacing each vertex by a coclique of order r . Based on this result we show, among other results, that a random tree with $m + 1$ edges a.a.s. decomposes the complete graph K_{6m+5} minus one edge.

Graph decomposition and Ringel's conjecture and Polynomial method

1. INTRODUCTION

Given two graphs H and G we say that H decomposes G if G is the edge-disjoint union of isomorphic copies of H . The following is a well-known conjecture of Ringel.

Conjecture 1 (Ringel [14]). *Every tree with m edges decomposes the complete graph K_{2m+1} .*

The conjecture has been verified by a number of particular classes of trees, see the extensive survey by Gallian [9]. By using the polynomial method, the conjecture was verified by Kézdy [11] for the more general class of so-called *stunted* trees. As mentioned by the author, this class is still small among the set of all trees.

In this paper a random tree with m edges is an unlabelled tree chosen uniformly at random among all the unlabelled trees with m edges. Drmota and the author [8] used structural results on random trees to show that asymptotically almost surely (a.a.s.) a random tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$, thus providing an approximate result to another decomposition conjecture by Graham and Haggkvist which asserts that $K_{m,m}$ can be decomposed by any given tree with m edges.

Results in this vein have been also recently obtained by Böttcher, Hladký, Piguet and Taraz [3], where the authors show that, for any $\epsilon > 0$ and any Δ , every family of trees with maximum degree at most Δ and at most $\binom{n}{2}$ edges in total packs into the complete graph $K_{(1+\epsilon)n}$ for every sufficiently large n .

Coming back to decompositions, or perfect packings, let $g(m)$ be the smallest integer n such that any tree with m edges decomposes the complete graph K_n . It was shown by Yuster [17] that $g(m) = O(m^{10})$ and the upper bound was reduced by Kézdy and Snevily [12] to $g(m) = O(m^3)$. Since $K_{2m,2m}$ decomposes the complete graph K_{8m^2+1} (see Snevily [16]), the above mentioned result on the decomposition of $K_{2m,2m}$ shows that $g(m) = O(m^2)$ asymptotically almost surely.

In this paper we prove that one can decompose almost complete graphs by random trees, getting much closer to the original conjecture of Ringel.

For positive integers n, r we denote by $K_n(r)$ the blow-up graph obtained from the complete graph K_n by replacing each vertex by a coclique with order r and joining every

pair of vertices which do not belong to the same coclique. Our main result is the following one.

Theorem 1. *For any m such that $p = 2m + 1$ is a prime, and every $r \geq 2$, asymptotically almost surely a random tree with m edges decomposes $K_{2m+1}(r)$.* \square

As an application of Theorem 1 we obtain the following corollaries, which are approximate results for random trees of Ringel's conjecture.

The following statement is a direct consequence of Theorem 1 with $r = 2$.

Corollary 1. *For every m such that $p = 2m + 1$ is a prime, asymptotically almost surely, a random tree with m edges decomposes $K_{4m+2} \setminus M$, where M is any perfect matching.* \square

Next Corollaries also follow from Theorem 1 with some additional work.

Corollary 2. *For every m such that $p = 2m + 1$ is a prime, a random tree with $m + 1$ edges a.a.s. decomposes $K_{6m+5} \setminus e$, where e is an edge of the complete graph.* \square

By using similar techniques as the ones involved in the proof of Corollary 2 the result can be extended to any odd $r \geq 3$.

Corollary 3. *For each odd number $r \geq 3$ and every m such that $p = 2m + 1$ is a prime a random tree with $m + 1$ edges a.a.s. decompose*

$$K_n \setminus K_t.$$

where $t = (r + 1)/2$ and $n = r(2m + 1) + t$. \square

This extension of Corollary 2 can be seen as an approximation to a more general conjecture by Ringel which states that every tree with m edges decomposes the complete graph K_{rm+1} whenever r and m are not both odd.

The paper is organised as follows. In Section 2 we introduce the notion of rainbow embeddings in connection to graph decompositions and give some results which provide a rainbow embedding of a given tree in an appropriate Cayley graph. The embedding techniques use the polynomial method of Alon and bring the condition that $p = 2m + 1$ is a prime in the statement of Theorem 1. These techniques are not enough to ensure that the rainbow embedded copy is isomorphic to the given tree. In order to complete the result we consider the blow up of the complete graph and perform some local modifications of the rainbow embedding in Section 3. The proofs of Theorem 1 and of the Corollaries 1, 2 and 3 are given in Section 4.

2. RAINBOW EMBEDDINGS

The general approach to show that a tree T decomposes a complete graph consists in showing that T cyclically decomposes the corresponding graph. We next recall the basic principle behind this approach in slightly different terminology.

A rainbow embedding of a graph H into an oriented arc-colored graph X is an injective homomorphism f of some orientation \vec{H} of H in X such that no two arcs of $f(\vec{H})$ have the same color. According to its common use, even if a rainbow embedding is meant to be defined as a map $f : V(H) \rightarrow V(X)$, we still call f the induced map $f : E(\vec{H}) \rightarrow E(X)$ on arcs defined as $f(x, y) = (f(x), f(y))$, and we think of f as a map $f : \vec{H} \rightarrow X$.

Let $X = \text{Cay}(G, S)$ be a Cayley digraph on an abelian group G with respect to an antisymmetric subset $S \subset G$ ($S \cap -S = \emptyset$.) We consider X as an arc-colored oriented graph, by giving to each arc $(x, x + s)$, $x \in G$, $s \in S$, the color s .

Lemma 1. *If a graph H admits a rainbow embedding in $X = \text{Cay}(G, S)$, where S is an antisymmetric subset of a group G of order n then the underlying graph of X contains n edge-disjoint copies of H . In particular, if H has $|S|$ edges then H decomposes the underlying graph of X .*

Proof. Let $f : H \rightarrow X$ be a rainbow embedding. For each $a \in G$ the translation $x \rightarrow x + a$, $x \in G$, is an automorphism of X which preserves the colors and has no fixed points. Therefore, each translation sends $f(\vec{H})$ to an isomorphic copy which is edge-disjoint from it. Thus the sets of translations for all $a \in G$ give rise to n edge-disjoint copies of \vec{H} in X . By ignoring orientations and colors, we thus have n edge disjoint copies of H in the underlying graph of X . In particular, if H has $|S|$ edges then H decomposes the underlying graph of X . \square

The proof of the main Theorem uses the above Lemma for a rainbow subgraph of an appropriate Cayley graph X . Instead of finding a rainbow embedding of a tree T we will find a rainbow edge-injective homomorphism of T in X in two steps, first embedding T_0 , the tree with some leafs removed, and then embedding the remaining forest F of stars to complete T .

For the first step we use the the so-called Combinatorial Nullstellensatz of Alon [1] that we next recall.

Theorem 2 (Combinatorial Nullstellensatz). *Let $P \in F[x_1, \dots, x_k]$ be a polynomial of degree d in k variables with coefficients in a field F .*

If the coefficient of the monomial $x_1^{d_1} \dots x_k^{d_k}$, where $\sum_i d_i = d$, is nonzero, then P takes a nonzero value in every grid $A_1 \times \dots \times A_k$ with $|A_i| > d_i$, for $1 \leq i \leq k$. \square

In the following Lemma we use Theorem 2 in a way inspired by Kézdy [11]. A *peeling ordering* of a tree T is an ordering x_0, \dots, x_m of $V(T)$ such that for every $0 \leq t \leq m$ the induced subgraph $T[x_0, \dots, x_t]$ is a subtree of T . We assume that T is an oriented tree with all its edges oriented from the root x_0 .

Next Lemma shows that any tree with k edges can be rainbowly embedded in some $\text{Cay}(\mathbb{Z}_p, S)$ with $|S| = k$ provided that k is not too large with respect to p .

Lemma 2. *Let $p > 10$ be a prime and T a tree with $k < 3(p-1)/10$ edges. There is an antisymmetric set $S \subset \mathbb{Z}_p^*$ with $|S| = k$ such that T admits a rainbow embedding in $\text{Cay}(\mathbb{Z}_p, S)$.*

Proof. Let x_0, x_1, \dots, x_k be a peeling ordering of T . Label the edges of T by variables y_1, \dots, y_k such that the edge labelled y_i joins x_i with $T[x_0, x_1, \dots, x_{i-1}]$, $0 < i \leq k$. For each i we denote by $T(0, i)$ the set of subscripts j such that the edges y_j lie in the unique path from x_0 to x_i in T . Consider the polynomial $P \in \mathbb{F}_p[y_1, \dots, y_k]$ defined as

$$P(y_1, \dots, y_k) = \prod_{1 \leq i < j \leq k} (y_j^2 - y_i^2) \prod_{1 \leq i < j \leq k} \left(\sum_{r \in T(0, i)} y_r - \sum_{s \in T(0, j)} y_s \right),$$

which has degree $2\binom{k}{2} + \binom{k}{2} = 3k(k-1)/2$.

Suppose that $P(a_1, a_2, \dots, a_k) \neq 0$ for some point $(a_1, \dots, a_k) \in (\mathbb{F}_p^*)^k$. Then, since the first factor $Q = \prod_{i < j} (y_i^2 - y_j^2)$ of P is nonzero at (a_1, \dots, a_k) , we have $a_i \neq \pm a_j$ for each pair $i \neq j$. Hence the set $S = \{a_1, \dots, a_k\}$ consists of pairwise distinct elements and it is antisymmetric.

Moreover, since the second factor $R = \prod_{i < j} (\sum_{y_r \in T(0,i)} y_r - \sum_{y_r \in T(0,j)} y_r)$ is nonzero at (a_1, \dots, a_k) , then the map $f : V(T) \rightarrow \text{Cay}(\mathbb{Z}_p, S)$ defined as $f(x_0) = 0$ and

$$f(x_i) = \sum_{r \in T(0,i)} y_r,$$

for $1 \leq i \leq k$, is injective and provides a rainbow embedding of T in $\text{Cay}(\mathbb{Z}_p, S)$.

Let us show that P is nonzero at some point of $(\mathbb{Z}_p^*)^k$. To this end we consider the monomial of maximum degree

$$y_k^{3(k-1)} y_{k-1}^{3(k-2)} \dots y_1^0,$$

in P . This monomial can be obtained in the expansion of P by collecting y_k in all the factors of Q where it appears, giving $y_k^{2(k-1)}$, and also in all terms of R where it appears, which, since y_k is a leaf of T , gives y_k^{k-1} . This is the unique way to obtain $y_k^{3(k-1)}$ in a monomial of P . Thus the coefficient of $y_k^{3(k-1)}$ in P is

$$[y^{3(k-1)}]P = \pm P_{k-1},$$

where

$$P_{k-1}(y_1, \dots, y_{k-1}) = \prod_{1 \leq i < j \leq k-1} (y_i^2 - y_j^2) \prod_{1 \leq i < j \leq k-1} \left(\sum_{r \in T(0,i)} y_r - \sum_{s \in T(0,j)} y_s \right).$$

By iterating the same argument we conclude that the coefficient in P of

$$y_k^{3(k-1)} y_{k-1}^{3(k-2)} \dots y_1^0$$

is ± 1 and, in particular, different from zero. Since $3(k-1) < 9p/10 < p-1$ for $p > 10$, we conclude from Theorem 2 that P takes a nonzero value in $(\mathbb{Z}_p^*)^k$. This concludes the proof. \square

In the second step we try to embed rainbowly a forest of stars. We still use Theorem 2, or rather the following consequence derived from it by Alon [2].

Theorem 3 (Alon [2]). *Let p be a prime and $k < p$. For every sequence a_1, \dots, a_k (possibly with repeated elements) and every set $\{b_1, \dots, b_k\}$ of elements of \mathbb{Z}_p there is a permutation $\sigma \in \text{Sym}(k)$ such that the sums $a_1 + b_{\sigma(1)}, \dots, a_k + b_{\sigma(k)}$ are pairwise distinct.* \square

The rainbow map defined with the help of Theorem 3 may fail to be a rainbow embedding of the forest because some endvertices may be sent to some center of another star. One consequence of the above result is that, for every antisymmetric set $S \subset \mathbb{Z}_p$ with h elements, every forest of stars with h edges admits a rainbow ‘quasi’ embedding in $\text{Cay}(\mathbb{Z}_p, S)$. Moreover, the centers of the stars in the forest can be placed at prescribed vertices. The following is the precise statement.

Lemma 3. *Let p be a prime. Let F be a forest of k stars centered at x_1, \dots, x_k and $h \leq (p-1)/2$ edges. Let $S \subset \mathbb{Z}_p^*$ be an antisymmetric set with $|S| = h$.*

Every injection $f : \{x_1, \dots, x_k\} \rightarrow \mathbb{Z}_p$ can be extended to a rainbow edge-injective homomorphism, $f_1 : F \rightarrow \text{Cay}(\mathbb{Z}_p, S)$ such that $f_1(F)$ is an oriented graph with maximum indegree one.

Proof. Consider the sequence $(f(x_1)^{h_1}, \dots, f(x_k)^{h_k})$, where the multiplicity h_i of $f(x_i)$ is the number of leafs of the star centered at x_i , $\sum_i h_i = h$.

By Theorem 3 there is a numbering s_1, \dots, s_h of the elements of S such that for any $1 \leq i \leq k$ and any $\sum_{r=1}^{i-1} h_r < j \leq \sum_{r=1}^i h_r$, the sums

$$f(x_i) + s_j,$$

are pairwise distinct.

Label the noncenter vertices of F by y_1, \dots, y_k , such that y_j adjacent to x_i whenever $\sum_{r=1}^{i-1} h_r < j \leq \sum_{r=1}^i h_r$, and orient the edges of F from the centers of the stars to their endvertices.

For each i and each $\sum_{r=1}^{i-1} h_r < j \leq \sum_{r=1}^i h_r$, we obtain the desired rainbow embedding by defining,

$$f_1(x_i) = f(x_i), \quad f_1(y_j) = f(x_i) + s_j.$$

Since all sums are distinct, no two endvertices of F are sent to the same vertex by f_1 and each of them has indegree one in $f_1(F)$; by the same reason, every $f_1(x_i)$ can coincide with at most one $f_1(y_j)$ for some y_j not in the same star as x_i . Thus the image $f_1(F)$ has indegree at most one. \square

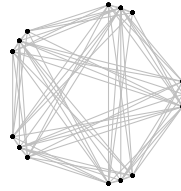
3. THE DECOMPOSITION

In this Section we prove Theorem 1. The strategy of the proof is as follows. We decompose the given tree T into a tree T_0 and a forest F of stars centred at some vertices of T_0 ,

$$T = T_0 \oplus F.$$

We embed T rainbowly in $\text{Cay}(\mathbb{Z}_p, S)$ where $S \subset \mathbb{Z}_p^*$ is an antisymmetric set of cardinality $|S| = (p-1)/2$. We embed T_0 and F by using Lemma 2 and Lemma 3. Doing so, the image of T by the rainbow embedding may be nonisomorphic to T . The last step in the proof consists in extending the rainbow embedding to $\text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$ and rearranging some arcs to obtain a decomposition of this oriented graph into copies of T .

For a graph G and a positive integer r we denote by $G(r)$ the graph obtained from G by replacing each vertex with a coclique of order r and every vertex is adjacent to all vertices except the ones in their coclique (below $K_5(3)$ illustrates the definition). The same notation is used when G is an oriented graph.



We will need the following technical Lemma.

Lemma 4. *Let $r \geq 2$ be an integer and let*

$$M = (M_a, M_b) = \left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & r & \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ r+1 & r+2 & \cdots & 2r & \sigma_{r+1} & \sigma_{r+2} & \cdots & \sigma_{2r} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ r(r-1)+1 & r(r-1)+2 & \cdots & r^2 & \sigma_{r(r-1)+1} & \sigma_{r(r-1)+2} & \cdots & \sigma_{r^2} \end{array} \right).$$

be a matrix where $(\sigma_1, \dots, \sigma_{r^2})$ is a permutation of $\{1, \dots, r^2\}$.

Then, there are permutations of the elements in each column of M in such a way the resulting matrix M' has no row with repeated entries.

Proof. We proceed row by row. By the definition of M , each column has r distinct entries. Let $M_{a,i}$ be the set of entries in the i -th column of M_a and $M_{b,j}$ be the set of entries in the j -th column of M_b .

We use Hall's theorem to find a transversal of the family of

$$\mathcal{M} = \{M_{a1}, M_{a2}, \dots, M_{ar}, M_{b1}, M_{b2}, \dots, M_{br}\}.$$

For each pair of subsets $I, J \subset \{1, 2, \dots, r\}$ we have,

$$(1) \quad |I| + |J| \leq 2 \max\{|I|, |J|\} \leq r \max\{|I|, |J|\} \leq |(\cup_{i \in I} M_{a,i}) \cup (\cup_{j \in J} M_{b,j})|$$

which shows that Hall's condition holds and therefore \mathcal{M} has a transversal. We place this transversal in the first row of the new matrix M' .

By deleting each element of the transversal from its set of \mathcal{M} we get a family of $(r-1)$ -sets for which the inequalities in (1) hold with r replaced by $(r-1)$ as long as $r-1 \geq 2$. Hence there is a transversal of this new family of sets which we place in the second row of M' . We can proceed with the same argument up to the $(r-1)$ row. Now if each of the first $r-1$ rows of M' have their entries pairwise distinct, the remaining elements are also pairwise distinct and can be placed in the last row of M' . \square \square

We next proceed to the main Lemma in this Section.

Lemma 5. *Let $p > 10$ be a prime and $r \geq 2$ an integer. Let T be a tree with $m = (p-1)/2$ edges and at least $2m/5$ leafs. Then T decomposes $K_{2m+1}(r)$.*

Proof. Remove $\lceil 2m/5 \rceil$ leaves from T and denote by T_0 the resulting tree. Let F be the forest of stars with centers in vertices of T_0 so that

$$T = T_0 \oplus F.$$

We split the proof of the Lemma into three steps.

Step 1. Define a rainbow embedding of T into $X = \text{Cay}(\mathbb{Z}_p, S)$ where $S \subset \mathbb{Z}_p^*$ is an antisymmetric set with $|S| = (p-1)/2$. By ignoring the orientations of X we obtain the complete graph K_p .

Let $t \leq 3m/5 < 3(p-1)/10 < (p-1)/3$ be the number of edges of T_0 . By Lemma 2, there is an antisymmetric subset $S_0 \subset \mathbb{Z}_p^*$ with $|S_0| = t$ and a rainbow embedding

$$f_0 : T_0 \rightarrow \text{Cay}(\mathbb{Z}_p, S_0).$$

Let x_0, \dots, x_t be a peeling ordering of T_0 . Since $t > \lceil 2m/5 \rceil$, we may assume that x_0 is not incident with a leaf in F . By exchanging elements of S by their opposite ones if necessary, we may assume that $f_0(T_0)$ has all its edges oriented from x_0 to the leaves of T_0 . By abuse of notation we still denote by x_0, \dots, x_t the images of the vertices of T_0 by f_0 . We may assume that $f_0(x_0) = 0$.

Let S be an antisymmetric subset of \mathbb{Z}_p^* with $|S| = (p-1)/2$ which contains S_0 , so that $|S - S_0| = |E(F)|$. Let $x_{i_1} = v_1, \dots, x_{i_k} = v_k$ be the centers of the stars of F . By Lemma 3 there is an edge-injective rainbow homomorphism of the forest F into $\text{Cay}(\mathbb{Z}_p, S \setminus S_0)$,

$$f_1 : F \rightarrow \text{Cay}(\mathbb{Z}_p, S \setminus S_0),$$

such that $f_1(v_i) = f_0(v_i)$, where v_1, \dots, v_k are the centers of the stars of F . Moreover $\tilde{F} = f_1(F)$ is an oriented graph with maximum indegree one.

The map $f : V(T) \rightarrow \text{Cay}(\mathbb{Z}_p, S)$ defined by f_0 on $V(T_0)$ and by f_1 on $V(F)$ is well defined, since $f_1(v_i) = f_0(v_i)$, and

$$f(T) = f_0(T_0) \oplus f_1(F) = H$$

is a rainbow subgraph of $X = \text{Cay}(\mathbb{Z}_p, S)$.

We note that f may fail to be a rainbow embedding of T in $X = \text{Cay}(\mathbb{Z}_p, S)$ to the effect that some endvertices of T may have been sent by f_1 to some vertices of $f_0(V(T_0))$. Thus H may be not isomorphic to T and contain some cycles (see Figure 1 for an illustration.)

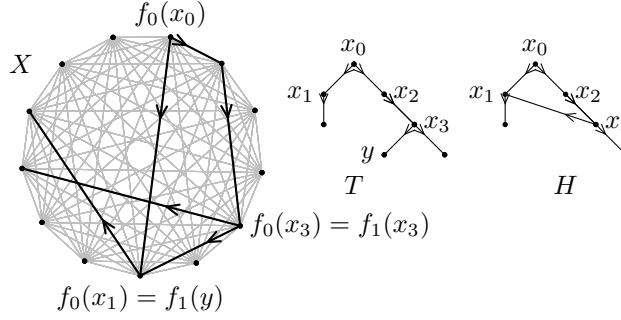


FIGURE 1. A rainbow map of T with conflicting arcs at $f_0(x_1) = f_1(y)$.

We observe however that, if $f_1(y) = f_0(x)$ for some endvertex $y \in F$ and some $x \in V(T_0)$, then y is not adjacent to x in T because f_1 is an edge-injective homomorphism. In other words, $f(T)$ has maximum indegree at most two.

Step 2. Extending the rainbow map to $X(r) = \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$.

Let $Y = X(r)$. By ignoring the orientations and colors of the arcs in Y we obtain $K_p(r)$.

For each pair $i, j \in \mathbb{Z}_r$ we define a subgraph H_{ij} of Y as the image by the injective homomorphism $f_{ij} : H \rightarrow Y$ such that $f_{ij}(0) = (0, i)$ and every arc $(x, x + s) \in E(H)$ is sent to the arc $(f_{ij}(x), f_{ij}(x) + (s, j))$ of $E(Y)$.

Since H is a connected subgraph of X , the map f_{ij} is well defined. Moreover $H_{ij} = f_{ij}(H)$ is a rainbow subgraph of Y . Below there is an illustration for $p = 13$ of the subgraphs $H_{0,j}$ corresponding to the example of Figure 1.

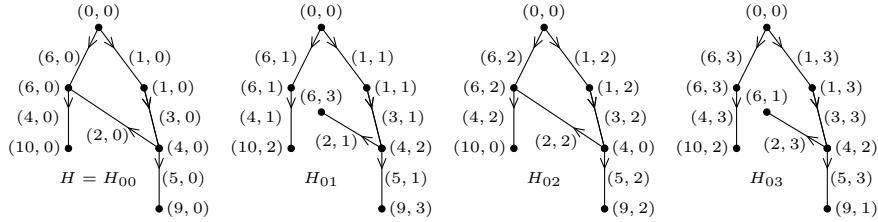


FIGURE 2. The rainbow subgraphs $H_{0,j} = f_{0j}(H)$ of $X(4) = \text{Cay}(\mathbb{Z}_{13} \times \mathbb{Z}_4, S \times \mathbb{Z}_4)$.

Every H_{ij} can be decomposed into

$$H_{ij} = T_{ij} \oplus F_{ij},$$

where, since T_0 is acyclic, T_{ij} is isomorphic to T_0 . As in the Step 1, H_{ij} may be non isomorphic to the original tree T , but only due to the fact that some end vertex of F_{ij} may

have been identified with some vertex of T_{ij} . However, the in-degree of every vertex in H_{ij} is again at most two as this was the case in H . If there is a vertex with indegree two in H_{ij} we call its incoming arcs to be *conflicting*.

We note that the H_{ij} 's are edge-disjoint (they hold pairwise distinct labels for j fixed and these labels emerge from distinct vertices for each i). Let

$$H_i = \oplus_{0 \leq j < r} H_{ij} \quad \text{and} \quad H(r) = \oplus_{0 \leq i < r} H_i = \oplus_{0 \leq i, j < r} H_{ij}.$$

By the definition of f_{ij} , we observe that each H_i is a rainbow subgraph of Y with $r(p-1)/2$ edges, so that all colors of the generating set $S \times \mathbb{Z}_r$ of Y appear in H_i precisely once.

Step 3. The final step consists of modifying each H_{ij} into H'_{ij} , which will be isomorphic to the original tree T , in such a way that,

$$H(r) = \oplus_{0 \leq i, j < r} H'_{ij}.$$

Each arc (x, y) in H is split in $H(r)$ into a (oriented) complete bipartite graph $K_{r,r}$ that we denote by $K_{r,r}^{(x,y)}$. The H'_{ij} will be constructed by rearranging the arcs in $K_{r,r}^{(x,y)}$ whenever y has indegree two in H . This rearrangement of arcs will be performed locally not affecting the remaining arcs of H_{ij} .

Suppose that $y = f_1(u)$, where $y \in V(T_0)$ and $u \in V(F)$, so that y is incident with a conflicting arc of H .

Let x be the vertex of T_0 adjacent to y in T_0 and let z be the vertex of F adjacent to y in H (which creates an undesired cycle; see Figure 3 for an illustration.)

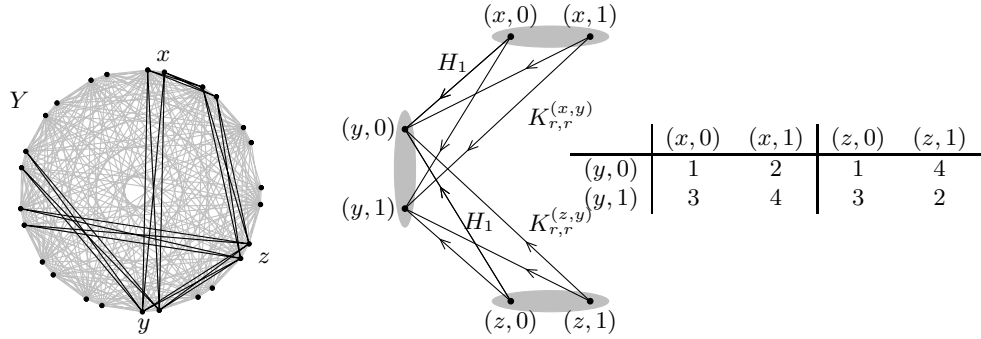


FIGURE 3. Conflicting arcs at y .

Each edge in $K_{r,r}^{(x,y)}$ belongs to one of r^2 trees T_{ij} isomorphic to T_0 in the decomposition of $H(r)$ and likewise, each edge in $K_{r,r}^{(z,y)}$ belongs to one of the F_{ij} . For simplicity we label these copies with the numbers $(i+1) + rj \in \{1, 2, \dots, r^2\}$.

We construct the $r \times r$ matrix M_x by placing at the entry $0 \leq i, j < r$ the label of the copy T_{ij} of T_0 which contains the arc $((x, i), (y, j))$. Likewise the $r \times r$ matrix M_z has the number $(i' + 1) + rj'$ in the entry (i, j) if $((z, i), (y, j))$ belongs to $F_{i'j'}$. Without loss of

generality we may assume that

$$(M_x, M_z) = \left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & r & \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ r+1 & r+2 & \cdots & 2r & \sigma_{r+1} & \sigma_{r+2} & \cdots & \sigma_{2r} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ r(r-1)+1 & r(r-1)+2 & \cdots & r^2 & \sigma_{r(r-1)+1} & \sigma_{r(r-1)+2} & \cdots & \sigma_{r^2} \end{array} \right),$$

for some permutation $\sigma = (\sigma_1, \dots, \sigma_{r^2})$ of $\{1, \dots, r^2\}$.

If all the rows of (M_x, M_z) have pairwise distinct entries, then no vertex in $y \times \mathbb{Z}_r$ belongs to a cycle of H_{ij} . Thus our goal is to redistribute the edges of $K_{r,r}^{(x,y)}$ and/or $K_{r,r}^{(z,y)}$ among the H_{ij} in such a way that the resulting matrix (M'_x, M'_z) has no rows with repeated entries.

Every permutation of the entries in one column of M_x and in one column of M_z does not affect the fact that we have an edge decomposition of $H(r)$.

By Lemma 4, there is a matrix $M' = (M'_x, M'_z)$ obtained from M by permuting the entries within columns which have no repeated entries in the same row. We assign the arcs to the numbered trees and forests according to the new matrix $M' = (M'_x, M'_z)$. By doing so, there are no conflicting arcs incident to vertices in $y \times \mathbb{Z}_r$.

This local rearrangement may have affected the connectivity of the copies of T_{ij} on the arcs coming out from vertices in $y \times \mathbb{Z}_r$. It may happen that some $H_{i,j}$ is no longer incident to the same vertex in $y \times \mathbb{Z}_r$ as the vertex incident from the next arc coming out in the same graph (see Figure 3 for an illustration.)

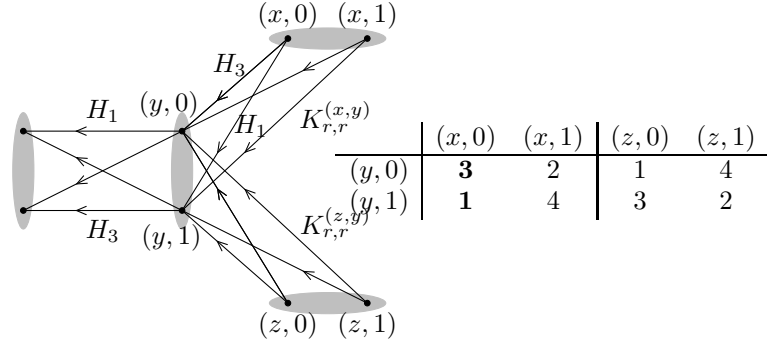


FIGURE 4. Distribution of arcs after rearrangement: H_1 and H_3 have been disconnected.

In order to repair the connectedness of the H_{ij} 's and, at the same time, do not affect these graphs outside the conflicting point in which rearranging of arcs was performed, we reassign the arcs coming out from vertices in $y \times \mathbb{Z}_r$ to copies of H_{ij} so that they originate where the rearrangement placed them and point to the same vertex they did before the arrangement. Such a local reassignment can be always achieved within the $K_{r,r}^{(y,y')}$ corresponding to every arc (y, y') in H .

We can make the local arrangements described above by following the original peeling order of T_0 . We proceed to modify the distribution of the arcs as we encounter vertices incident with conflicting arcs in that order. In this way we travel through directed arcs from the root of each T_{ij} , so that rearrangements of arcs do not affect modifications made previously until all conflicting arcs have been processed.

At this point we obtain an edge decomposition of $H(r)$ into the r^2 oriented graphs $H'_{i,j}$, each one isomorphic to our given tree T . What remains is to use this decomposition of $H(r)$ to produce a decomposition of the whole Cayley graph Y .

Let

$$H'_i = \oplus_{0 \leq j < r} H'_{ij}.$$

Since H_i is a rainbow subgraph of Y , the construction of H'_i yields also a rainbow subgraph of Y . Therefore, the set of translates

$$\{H'_i + (x, 0) : x \in \mathbb{Z}_p\}$$

is edge-disjoint. On the other hand, this set of translates $\{H'_i + (x, 0) : x \in \mathbb{Z}_p\}$ is vertex disjoint with $\{H'_{i'} + (x, 0) : x \in \mathbb{Z}_p\}$ with $i' \neq i$. In particular, both graphs are also edge-disjoint. Thus

$$Y = \oplus_{x \in \mathbb{Z}_p} (\oplus_{0 \leq i < r} H'_i + (x, 0)) = \oplus_{x \in \mathbb{Z}_p} ((\oplus_{0 \leq i, j < r} H'_{ij}) + (x, 0))$$

is a decomposition of Y into copies of T . This completes the proof. \square \square

4. PROOFS OF THE RESULTS

Lemma 5 leads directly to a proof of Theorem 1.

Proof of Theorem 1. Robinson and Schwenk [15] proved that the average number of leafs in an (unlabelled) random tree with m edges is asymptotically cm with $c \approx 0.438$. Drmota and Gittenberger [6] showed that the distribution of the number of leafs in a random tree with m edges is asymptotically normal with variance $c_2 m$ for some positive constant c_2 . Thus, asymptotically almost surely a random tree with m edges has more than $2m/5$ leafs. It follows from Lemma 5 that a tree with at least $2m/5$ leafs decomposes $K_{2m+1}(r)$ for each $r \geq 2$ and $m = (p-1)/2 \geq 5$ edges, where $p > 10$ is a prime. \square

Corollary 1 follows from Theorem 1 with $r = 2$, because $K_{2m+1}(2)$ is isomorphic to $K_{4m+2} \setminus M$, for M any matching of K_{4m+2} . \square

We next prove Corollary 2.

Proof of Corollary 2. Let T be a random tree with $m+1$ edges. Let T' be the tree obtained from T by deleting one leaf yz , where z is an end vertex of T . As explained in the proof of Theorem 1, we know that (a.a.s.) the tree T' has at least $\frac{2m}{5}$ end vertices.

In what follows we use the notation from the proof of Lemma 5. Consider the decomposition of the Cayley graph $Y = \text{Cay}(\mathbb{Z}_{2m+1} \times \mathbb{Z}_3, S \times \mathbb{Z}_3)$ into copies of T' (with its arcs oriented from the root to the leafs of a peeling ordering of T') which is described in that proof. It remains to complete each copy to T by adding the missing leaf.

To this end we add two additional vertices α, β to Y and make them adjacent from every vertex in Y . Moreover we add to Y an oriented triangle in each stable set of Y . The resulting graph Y' (omitting orientations and colours) is isomorphic to $K_{6m+5} \setminus e$, where $e = \{\alpha, \beta\}$. We next describe how to complete each copy of T' to $T = T' + yz$. We consider two cases.

Suppose first that y is not incident to a conflicting arc in $H = f(T')$. In this case each of $(y, 0), (y, 1), (y, 2)$ has indegree three in $H(3)$. We assign the three outgoing arcs added to Y from each (y, j) to one of its three incoming trees bijectively. By repeating this procedure to each translate $H(3) + (z, 0)$, $z \in \mathbb{Z}_p$, in Y we obtain a decomposition of $K_{6m+5} \setminus e$, into copies of T . This completes the proof in this case.

Suppose now that y is incident to a conflicting arc in $H = f(T')$. Since in this case y has indegree two in H , each vertex (y, j) has indegree six in $H(3)$ and, by the construction of the H'_{ij} , the six incoming arcs belong to six different trees (this was actually the purpose in the construction of H'_{ij} .) There are nine trees in total incident to the three vertices $(y, 0), (y, 1), (y, 2)$ in $H(3)$, label them T'_1, \dots, T'_9 . Following the notation from the proof of Lemma 5, we may assume that the row j of the matrix

$$(M_x, M_z) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & \sigma_1 & \sigma_2 & \sigma_3 \\ 4 & 5 & 6 & \sigma_4 & \sigma_5 & \sigma_6 \\ 7 & 8 & 9 & \sigma_7 & \sigma_8 & \sigma_9 \end{array} \right),$$

denotes the labels of the trees incident with (y, j) , where $\sigma = (\sigma_1, \dots, \sigma_9)$ is a permutation of $\{1, \dots, 9\}$ and each row has no repeated entries (as it was shown in the proof of Lemma 5.) Figure 5 illustrates the situation.

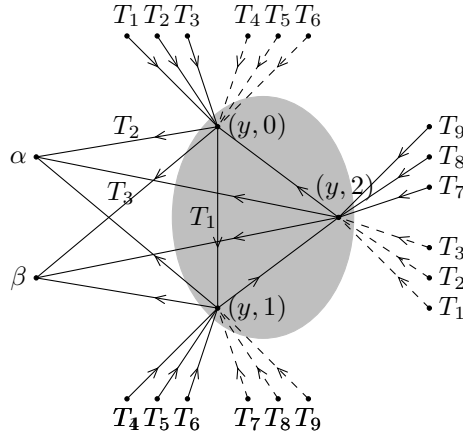


FIGURE 5. Dotted lines indicate the trees labeled in the right matrix M_z . A good assignment on $(y, 0)$ is shown in the picture. Only one of the two orientations of the triangle admits a good assignment in this example.

Consider the orientation $(y, 0), (y, 1), (y, 2)$ of the triangle induced by these three vertices in Y' . We assign the three outgoing arcs added to Y from each (y, j) to one of its three incoming trees in row j of M_x bijectively. There is one such bijection which avoids creating a cycle with the incoming trees at $(y, j + 1)$ unless the same trees are listed in row $j + 1 \pmod{3}$ of M_z , in which case we can instead create a cycle. In this situation, up to a permutation, the matrix (M_x, M_z) would have the form

$$(M_x, M_z) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & \sigma_1 & \sigma_2 & \sigma_3 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 7 & 8 & 9 & \sigma_7 & \sigma_8 & \sigma_9 \end{array} \right),$$

Then, the reverse orientation $(y, 0), (y, 2), (y, 1)$ admits a good assignment avoiding undesired cycles. Indeed, if the entries of some row j are the same as the entries of row $j - 1 \pmod{3}$ of the above matrix, then the remaining row would contain the same entries in M_x as the ones in M_z ,

$$(M_x, M_z) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{array} \right),$$

contradicting that the matrix has no repeated entries in each row. This completes the proof. \square

The argument used in the above proof can be extended to prove Corollary 3.

Proof of Corollary 3: We imitate the proof of Corollary 2. Choose an end vertex y of T and delete the leaf xy so that the resulting tree T' has m edges and at least $2m/5$ end vertices. By Lemma 5 we obtain a decomposition of $Y = \text{Cay}(\mathbb{Z}_{2m+1} \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$ by copies of an orientation of T' .

Consider the oriented graph Y' obtained from Y by adding $(r+1)/2$ new vertices $\alpha_1, \dots, \alpha_{(r+1)/2}$ and all arcs from Y to these vertices. Moreover we insert a regular tournament T_r in each stable set of Y . By removing the orientations, Y' is isomorphic to $K_{r(2m+1)+\frac{r+1}{2}} \setminus K_{(r+1)/2}$ (the vertices form a stable set in Y' .)

Each vertex x in $H(r)$ is incident to r copies of T' if x is adjacent to y in T .

We next add one leaf to each copy of T' by using the $(r+1)/2$ arcs to $\alpha_1, \dots, \alpha_{(r+1)/2}$ and the $(r-1)/2$ arcs in the regular tournament through that vertex. This results in r copies of T in Y' if there were not conflicting arcs of the oriented graph H used to obtain the decomposition of Y . If this was not the case, then there is some regular tournament which admits a good assignment in the sense described in the proof of Corollary 2. We omit the details of this last statement. \square

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